Section 1-5, Mathematics 108

Solving Equations

Linear Equations

$$4x + 7 = 19$$
$$4x + 7 - 7 = 19 - 7$$
$$4x = 12$$
$$4x \cdot \frac{1}{4} = 12 \cdot \frac{1}{4}$$
$$x = 3$$

Quadratic Equations

1) Try Factoring

2) Completing the square?

Recall that

$$A^2 \pm 2AB + B^2 = \left(A \pm B\right)^2$$

That means that an equation

$$ax^2 + bx + c = 0$$

We can create a perfect square from the first two terms by making the third term

$$\left(\frac{b}{2a}\right)^2$$

Example:

$$x^2 - 8x + 13 = 0$$

We find $\left(\frac{8}{2\cdot 1}\right)^2 = 4^2 = 16$

$$x^2 - 8x + 16 + 13 = 16$$

$$\left(x-4\right)^2 + 13 = 16$$

$$\left(x-4\right)^2 = 3$$

$$x - 4 = \pm\sqrt{3}$$

$$x = 4 \pm \sqrt{3}$$

$$x = 4 + \sqrt{3}, 4 - \sqrt{3}$$

Quadratic Formula

We can apply this procedure to the general equation and get the quadratic formula

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} = \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note the discriminant $b^2 - 4ac$ tells us about the roots or zeros.

$b^2 - 4ac = 0 \quad 1 \quad \textit{solution}$

$b^2 - 4ac > 0 \quad 2 \quad \textit{solutions}$

$b^2 - 4ac < 0 \quad 0 \quad \textit{solutions}$

Equations with radicals (can be tricky)

Example:

$$2x = 1 - \sqrt{2-x}$$

First step, get rid of the radical by squaring

$$2x = 1 - \sqrt{2-x}$$
$$\sqrt{2-x} = 1 - 2x$$
$$2 - x = (1-2x)^2$$
$$2 - x = 1 - 4x + 4x^2$$
$$4x^2 - 3x - 1 = 0$$

Using the quadratic formula

$$x = \frac{3 \pm \sqrt{9+16}}{8} = \frac{3 \pm \sqrt{25}}{8} = \frac{3 \pm 5}{8} = 1, -\frac{1}{4}$$

However squaring can introduce "phantom" roots, so we need to plug these roots back in to check.

$$2(1) = 1 - \sqrt{2-1}$$
$$2 = 1 - \sqrt{1}$$
$$2 = 0$$

So 1 is a phantom root, not a solution.

$$2\left(\frac{-1}{4}\right) = 1 - \sqrt{2 - \frac{-1}{4}}$$
$$-\frac{1}{2} = 1 - \sqrt{\frac{9}{4}} = 1 - \frac{3}{2} = -\frac{1}{2}$$

So $-\frac{1}{4}$ is a solution.

Simple Higher Degree Equations

Sometimes a 3rd or 4th degree polynomial equation is really a quadratic in disguise.

Example:

$$x^4 - 8x^2 + 8 = 0$$

In this equation let $y = x^2$ so

$$y^2 - 8y^2 + 8 = 0$$
$$y = \frac{8 \pm \sqrt{64 - 32}}{2} = 4 \pm 2\sqrt{2}$$
$$x = \pm\sqrt{4 \pm 2\sqrt{2}}$$

Using all 4 combinations of + an - you get 4 distinct roots or solutions.

Fractional Powers

Example:

$$x^{1/3} + x^{1/6} - 2 = 0$$

Let $y = x^{1/6}$

$$y^2 + y - 2 = 0$$
$$y = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$$
$$y^3 = x$$
$$x = 1, -8$$

Checking the roots we find that 1 is a solution, but -8 is a phantom.

Absolute Value Equations

When you have an absolute value in an equation you really have two different equations. Split it up and solve each individually

Example:

Try them both out

$$|2 \cdot 4 - 5| = 3 \qquad |2 \cdot 1 - 5| = 3$$

$$|3| = 3 \qquad |-3| = 3$$

$$3 = 3 \qquad 3 = 3$$

Supplemental (You don't need to know this)

Since there is a quadratic formula, is there a similar solution for polynomial equations of the third degree, a cubic equation?

Yes!

Niccolò Fontana Tartaglia, an Italian mathematician who lived from 1499-1557 came up with this formula.

Given the equation $ax^3 + bx^2 + cx + d = 0$

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$- \frac{b}{3a}$$

Similarly there is a quartic formula discovered by Lodovico Ferrari in 1540.

The search for a formulaic solution to the general 5th degree equation went on for almost 300 years until Niels Henrik Abel, a Norwegian mathematican showed in 1820 that no such formula could exist.

The search for this formula served to develop what is known today as the subject "Modern Algebra", a course you might take as an undergraduate mathematics major after Calculus and Linear Algebra.